Approximate Composition

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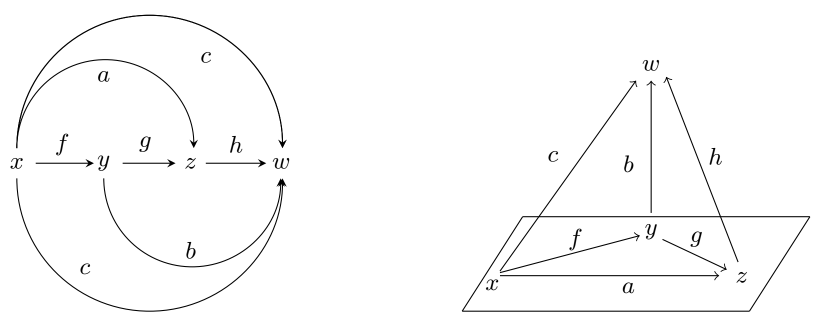
This talk is as much about the absence of a composition law as it is about the presence of such. Its core part generalizess the Aliouche-Simpson [1] notion of *approximate categorical structure* that is based on a directed graph in which one can measure the "distance" between a pair of consecutive arrows and any candidate for their composite arrow. Geometrically one may think of this "distance" as the "area" of the directed triangle spanned by the three arrows in question.

Here is an "every-day" example that may demonstrate the need for considering such a structure. Anybody searching for a flight from x to z ("vertices, objects") will have no difficulty comparing the prices of all *direct* flights ("edges, morphisms") from x to z. But how do we compare these with potentially cheaper *connecting* flight options, via various intermediate locations y? A mere price comparison doesn't take into account the loss of time, comfort, risk of missing the connection or losing one's luggage, *etc*, when one chooses a connecting flight option over a direct one. In modelling this situation it does not suffice to provide just a metric on the set of edges from x to z, even if it somehow cooperates with the concatenation of edges. Rather, a more appropriate model should come equipped with a "distance" function $\delta(f, g, h)$ which will provide a comparison between a direct flight $h : x \longrightarrow z$ and a pair of consecutive connecting flights $f : x \longrightarrow y$, $g : y \longrightarrow z$.

We propose that a crucial condition on such a distance function should be a directed version of the *tetrahedral inequality* required by Gähler [2] for his *2-metrics* which, in turn, is a specialization of the *simplex inequality* required earlier by Menger [4] for his *n-metrics*. Aliouche and Simpson interpreted the tetrahedral inequality (whereby the area of one side of a tetrahedron cannot exceed the sum of the areas of the other three sides) as a substitute for the associativity of the missing composition:



$$\delta(f, g, a) \otimes \delta(g, h, b) \otimes \delta(f, b, c) \geq \delta(a, h, c) \quad \text{(left associativity law)}$$
$$\delta(f, g, a) \otimes \delta(g, h, b) \otimes \delta(a, h, c) \geq \delta(f, b, c) \quad \text{(right associativity law)}$$

The notion of *metric approximate category* as presented in [7], *metágory* for short, should be seen as a first step towards a Lawvere-style [3] generalization of the Aliouche-Simpson notion: we let "distances" live in a commutative unital quantale $(\mathcal{V}, \leq, \otimes, \mathsf{k})$, the prototype of which is the extended additive real half-line $\mathbb{R}^+ = ([0, \infty], \geq, +, 0)$ (or, isomorphically, the multiplicative unit interval $([0, 1], \leq, \cdot, 1)$).

The symmetric monoidal-closed category $\mathcal{V}$ gives rise to a number of further symmetric monoidal-closed categories:

- $\mathsf{Met}_{\mathcal{V}}$: $\mathcal{V}$-valued metric spaces (= small symmetric $\mathcal{V}$-categories) and their contractive maps;
- $\mathsf{Met}_{\mathcal{V}}$-$\mathsf{Gph}$: $\mathcal{V}$-metrically enriched graphs and their contractive graph homomorphisms;
- $\mathsf{Met}_{\mathcal{V}}$-$\mathsf{Cat}$: $\mathcal{V}$-metrically enriched small categories and their contractive functors;

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- $\mathsf{Metag}_{\mathcal{V}}$: $\mathcal{V}$-metagories and their contractors; this category sits between the previous two categories: every $\mathcal{V}$-metrically enriched category induces a $\mathcal{V}$-metagory which, in turn, induces a $\mathcal{V}$-metrically enriched graph.

The main statement here is that $\mathsf{Metag}_{\mathcal{V}}$ is symmetric monoidal-closed and, therefore, enriched in itself. In this way it may be regarded as the prototype of a *$\mathcal{V}$-valued approximate 2-category* since, while only admitting an approximate vertical structure, it does have a well-behaved horizontal composition law (to be outlined in [6]). The internal hom of $\mathsf{Metag}_{\mathcal{V}}$ was used in [7] to establish a Yoneda embedding of a so-called transitive $\mathcal{V}$-metagory into a $\mathcal{V}$-metrically enriched category. This special type of $\mathcal{V}$-metagory will be discussed along with other subtypes of $\mathcal{V}$-metagories that appear to be important. One of these subtypes (in the numerical context $\mathcal{V} = \mathbb{R}^+$) concerns those metagories which, for a fixed margin $\varepsilon > 0$, guarantee for every pair $f : x \longrightarrow y$, $g : y \longrightarrow z$ the existence of an arrow $h : x \longrightarrow z$ with $\delta(f, g, h) \leq \varepsilon$.

In [5], Pavlovic presented a *quantitative* framework for *formal concept analysis*, with a hands-on demonstration of the completion theory that he developed for the objects of his model. In the paper [6] we aim to extend his work to metagories in a similar vain, based on the Yoneda embedding of a metagory.

References

[1] A. Aliouche and C. Simpson: Approximate categorical structures. *Theory and Applications of Categories* 32(49) (20):1522–1562, 2017.

[2] S. Gähler: 2-metrische Räume und ihre topologische Struktur. *Mathematische Nachrichten* 26:115–148, 1963.

[3] F.W. Lawvere: Metric spaces, generalized logic, and closed categories. *Rendiconti del Seminario Matematico e Fisico di Milano* 43:135–166,1973. Author-commented version available in: *Reprints in Theory and Applications of Categories* 1:1–37, 2002.

[4] K. Menger: Untersuchungen über allgemeine Metrik. *Mathematische Annalen* 100:75-163, 1928.

[5] D. Pavlovic: Quantitative Concept Analysis. arXiv:1204.5802v1 [cs.LG].

[6] W. Tholen: Approximate composition. In preparation.

[7] W. Tholen and J. Wang: Metagories. *Topology and Its Applications*, to appear. Available at arXiv:1810.08828v2 [math.CT].